# CS 59300 – Algorithms for Data Science

## Classical and Quantum approaches

### Lecture 10 (10/07)

### Sum-of-Squares (III)

https://ruizhezhang.com/course_fall_2025.html

The slides is partly based on Ojas Parekh's talk

# What is quantum optimization?

| Algorithm \ Problem | Classical | Quantum |
| --- | --- | --- |
| Classical | Classical optimization | Classical approaches for quantum Hamiltonians (DMRG, mean-field methods, everything else) |
| Quantum | Quantum approaches for discrete optimization (AQC, QAOA for quantum Hamiltonians)<br>Quantum approaches for continuous optimization | Quantum approaches for quantum Hamiltonians (e.g. AQC, QAOA for quantum Hamiltonians) |

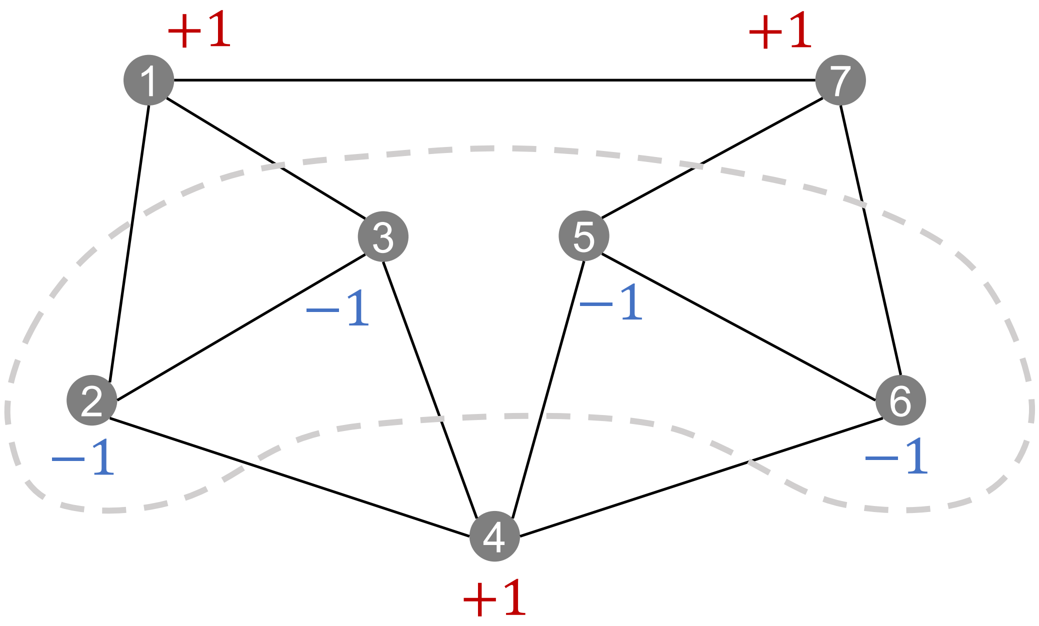
# (Classical) Max Cut

Given a graph $G(V, E)$, find a partition $f: V \to \{+1, -1\}$ maximizing

$$\sum_{ij \in E} \left( \frac{1 - f(i)f(j)}{2} \right)$$

"$f(i) \neq f(j)$"

- One of Karp's 21 **NP**-complete problems
- 0.878-approximation by Goemans and Williamson using SDP and randomized rounding




Max-cut = 8

# Interlude: Approximation algorithm

An $\alpha$-approximation algorithm runs in polynomial time, and for any instance $I$, delivers an approximate solution such that:

$$\frac{\text{Value}(\text{Approximation}_I)}{\text{Value}(\text{Optimal}_I)} \geq \alpha$$

Approximation algorithm = Relaxation + Rounding

- The approximation ratio can be lower bounded by:

$$\rho \coloneqq \min_I \frac{\text{Value}(\text{Approximation}_I)}{\text{fValue}(\text{Relaxation}_I)} \leq \alpha$$

- Integrality gap (barrier of the specific relaxation proof)

$$\min_I \frac{\text{Value}(\text{Optimal}_I)}{\text{fValue}(\text{Relaxation}_I)} \geq \rho$$

# Goemans-Williamson algorithm

## Relaxation:

For each vertex $i \in V$, assign $v_i \in \mathbb{R}^d$

$$\max \quad \sum_{(i,j)\in E} (1 - \langle v_i, v_j \rangle)/2$$
$$s.t. \quad \|v_i\| = 1 \ \ \forall i \in V$$

## Solving SDP:

$$\max \quad \langle -A_G/2\,, X \rangle$$
$$s.t. \quad X_{ii} = 1 \ \ \forall i \in V$$
$$X \succcurlyeq 0$$

## Cholesky decomposition:

$$X = \begin{bmatrix} - & v_1^\top & - \\ - & v_2^\top & - \\ & \vdots & \\ - & v_n^\top & - \end{bmatrix} \cdot \begin{bmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{bmatrix}$$

## Gaussian rounding:

- Sample a unit vector $g \in \mathbb{R}^d$
- $\sigma_i \leftarrow \mathrm{sign}(\langle g, v_i \rangle) \ \ \forall i \in V$

# (Classical) Max Cut

Given a graph $G(V, E)$, find a partition $f: V \to \{+1, -1\}$ maximizing

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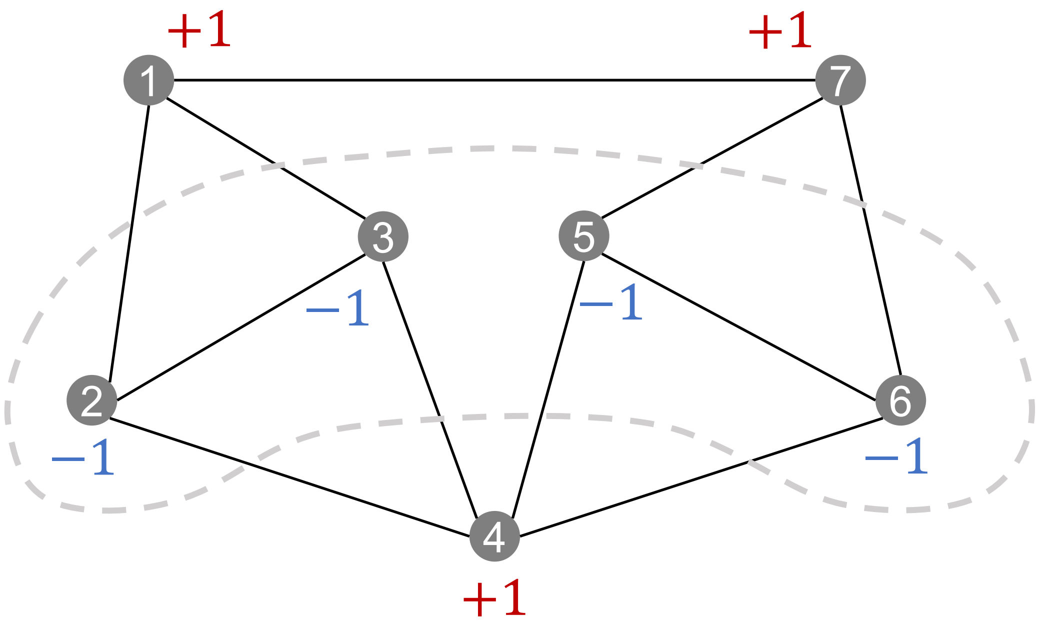


Max-cut = 8

What is the quantum version of Max Cut?

# Pauli matrices

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \qquad Y = \begin{bmatrix} 0 & -\mathbf{i} \\ \mathbf{i} & 0 \end{bmatrix}, \qquad Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

- $\{I, X, Y, Z\}$ is a basis for $2 \times 2$ Hermitian matrices
- $X^2 = Y^2 = Z^2 = I$
- Commutator and anticommutator: $[A, B] \coloneqq AB - BA$ and $\{A, B\} \coloneqq AB + BA$

$$[X, Y] = 2\mathbf{i}Z, \qquad [Y, Z] = 2\mathbf{i}X, \qquad [Z, X] = 2\mathbf{i}Y$$

$$\{X, Y\} = \{Y, Z\} = \{Z, X\} = 0$$ “swap flips the sign”

- Each of $X$, $Y$, $Z$ has one eigenvalue $+1$ and one eigenvalue $-1$ (their eigenvectors are called $X$-basis, $Y$-basis, $Z$-basis)

$X$-basis: $\quad |+\rangle = \dfrac{|0\rangle + |1\rangle}{\sqrt{2}}, \qquad |-\rangle = \dfrac{|0\rangle - |1\rangle}{\sqrt{2}}$

# Pauli matrices (multiple qubits)

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \qquad Y = \begin{bmatrix} 0 & -\mathbf{i} \\ \mathbf{i} & 0 \end{bmatrix}, \qquad Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

- For an $n$-qubit system, we use $\sigma_i$ for $\sigma \in \{X, Y, Z\}$ to denote applying $\sigma$ to the $i$-th qubit:

$$I \otimes \cdots \otimes I \otimes \sigma \otimes I \otimes \cdots \otimes I \in \mathbb{C}^{2^n \times 2^n}$$

## Pauli polynomial

- A monomial $\tau = \sigma_1 \sigma_2 \cdots \sigma_n$ with $\sigma_i \in \{I, X_i, Y_i, Z_i\}$
- $\deg(\tau) = |\{i \in [n] : \sigma_i \neq I\}|$
- $\mathcal{P}_n(k)$ is the set of monomials of degree at most $k$
- A Pauli polynomial is a real linear combination of monomials; its degree is the maximal degree over its terms

Hermitian operator

# Quantum Max-Cut (QMC)

- Let $G = (V, E)$ be a graph
- QMC is a special case of 2-local Hamiltonian problem:

Compute $\color{red}{\lambda_{\max}(H)}$ $\qquad H = \sum_{ij\in E} h_{ij}\,, \qquad \text{where} \quad h = \frac{1}{4}\cdot(I - XX - YY - ZZ)$

$$= \frac{1}{2}\cdot(|01\rangle - |10\rangle)(\langle 01| - \langle 10|)$$

$$\begin{array}{c} \\ 00 \\ 01 \\ 10 \\ 11 \end{array} \begin{array}{c} \begin{array}{cccc} 00 & 01 & 10 & 11 \end{array} \\ \begin{bmatrix} 0 & & & \\ & \color{red}{1} & -1 & \\ & -1 & \color{red}{1} & \\ & & & 0 \end{bmatrix} \end{array}$$

$|1\rangle$ $\quad$ $|0\rangle$

$|0\rangle$ $\quad$ or $\quad$ $|1\rangle$

(classical) Max-Cut

# Quantum Max-Cut (QMC)

- Let $G = (V, E)$ be a graph
- QMC is a special case of 2-local Hamiltonian problem:

Compute $\lambda_{\max}(H)$ $\quad H = \sum_{ij \in E} h_{ij}$, $\quad$ where $h = \frac{1}{4} \cdot (I - XX - YY - ZZ)$

$$= \frac{1}{2} \cdot (|01\rangle - |10\rangle)(\langle 01| - \langle 10|)$$

$$\frac{|01\rangle - |10\rangle}{\sqrt{2}}$$

quantum Max-Cut

"Entangled assignment" gets max value

$$\begin{array}{c c} & \begin{array}{cccc} 00 & 01 & 10 & 11 \end{array} \\ \begin{array}{c} 00 \\ 01 \\ 10 \\ 11 \end{array} & \begin{bmatrix} 0 & & & \\ & 1 & -1 & \\ & -1 & 1 & \\ & & & 0 \end{bmatrix} \end{array}$$

# Quantum Max-Cut (QMC)

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$$= \frac{1}{2} \cdot (|01\rangle - |10\rangle)(\langle 01| - \langle 10|)$$

**Term 1:** does nothing

quantum Max-Cut

# Quantum Max-Cut (QMC)

- Let $G = (V, E)$ be a graph
- QMC is a special case of 2-local Hamiltonian problem:

Compute $\lambda_{\max}(H)$ $\quad H = \sum_{ij \in E} h_{ij}\,,$ $\quad$ where $h = \frac{1}{4} \cdot (I \boxed{- XX} - YY - ZZ)$

$$= \frac{1}{2} \cdot (|01\rangle - |10\rangle)(\langle 01| - \langle 10|)$$


quantum Max-Cut

**Term 1:** does nothing

**Term 2:** measure in $X$ basis

- $-1$ if same ($+ +$ or $- -$)
- $+1$ if same ($+ -$ or $- +$)

# Quantum Max-Cut (QMC)

- Let $G = (V, E)$ be a graph
- QMC is a special case of 2-local Hamiltonian problem:

Compute $\lambda_{\max}(H)$ $\quad H = \sum_{ij \in E} h_{ij}\,,\qquad$ where $\;h = \frac{1}{4} \cdot (I \boxed{- XX} - YY - ZZ)$

$$= \frac{1}{2} \cdot (|01\rangle - |10\rangle)(\langle 01| - \langle 10|)$$


quantum Max-Cut

**Term 1:** does nothing

**Term 2:** should be different in $X$-basis

# Quantum Max-Cut (QMC)

- Let $G = (V, E)$ be a graph
- QMC is a special case of 2-local Hamiltonian problem:

Compute $\lambda_{\max}(H)$ $\quad H = \sum_{ij \in E} h_{ij}\,, \qquad$ where $\quad h = \frac{1}{4} \cdot (I - XX \boxed{- YY} - ZZ)$

$$= \frac{1}{2} \cdot (|01\rangle - |10\rangle)(\langle 01| - \langle 10|)$$


quantum Max-Cut

**Term 1:** does nothing

**Term 2:** should be different in $X$-basis

**Term 3:** should be different in $Y$-basis

# Quantum Max-Cut (QMC)

- Let $G = (V, E)$ be a graph
- QMC is a special case of 2-local Hamiltonian problem:

Compute $\lambda_{\max}(H)$ $\quad H = \sum_{ij\in E} h_{ij}$, $\quad$ where $h = \frac{1}{4}\cdot\left(I - XX - YY \boxed{- ZZ}\right)$

$$= \frac{1}{2}\cdot\left(|01\rangle - |10\rangle\right)\left(\langle 01| - \langle 10|\right)$$


quantum Max-Cut

Like (classical) Max-Cut in $X$, $Y$, and $Z$ basis

**Term 1:** does nothing

**Term 2:** should be different in $X$-basis

**Term 3:** should be different in $Y$-basis

**Term 4:** should be different in $Z$-basis

# Interlude: Quantum Lasserre hierarchy (ncSoS)

- Also called non-commutative sum-of-squares hierarchy
- Introduced by Navascués, Pironio, and Acin (NPA hierarchy)

Pseudo-density

- A $k$-positive pseudo-density $\tilde{\rho} \in \mathbb{C}^{2^n \times 2^n}$ is a $2^n \times 2^n$ Hermitian matrix
- $\mathrm{tr}[\tilde{\rho}] = 1$
- $\mathrm{tr}[\tilde{\rho} P^2] \geq 0,\ \forall$ Pauli polynomial $P$ of degree $\leq k$

We use $\widetilde{\mathcal{D}}_n(k)$ to denote the set of $k$-positive pseudo-density operators

- Level $k$ of the quantum Lasserre hierarchy finds an optimal $k$-positive pseudo-density matrix:

$$v_k(H) \coloneqq \max_{\tilde{\rho} \in \widetilde{\mathcal{D}}_k(n)} \mathrm{tr}[H\tilde{\rho}] \geq \lambda_{\max}(H)$$

- Convergence: $v_k(H) \geq v_{k+1}(H) \geq \cdots \geq v_n(H) = \lambda_{\max}(H)$

"tighter and tighter upper-bound"

# Quantum Lasserre hierarchy

- Let $\tilde{\rho}$ be the optimal pseudo-density solution to $\mathcal{L}_k$ (level $k$ Quantum Lasserre)
- For each Pauli monomial $\tau$, define its relaxed value to be

$$\langle \tau \rangle \coloneqq \mathrm{tr}[\tilde{\rho}\tau]$$

"pseudoexpectation"

- For QMC, $v_k(H)$ can be written as:

$$v_k(H) = \sum_{ij \in E} \frac{1}{4}\left(1 - \langle X_i X_j \rangle - \langle Y_i Y_j \rangle - \langle Z_i Z_j \rangle\right)$$

# Solve quantum Lasserre hierarchy

## Pseudoexpectation program

$$v_k(H) \coloneqq \max \sum_{\phi \in \mathcal{P}_n(2k)} c(\phi)\langle\phi\rangle$$

*Variables:* $\{\langle\tau\rangle : \tau \in \mathcal{P}_n(2k)\}$

*Constraints:*

- $\langle I \rangle = 1$
- $\mathcal{M}_k \in \mathbb{C}^{n^{\mathcal{O}(k)} \times n^{\mathcal{O}(k)}} : \mathcal{M}_k(\sigma,\tau) \coloneqq \langle\sigma\tau\rangle$ for any $\sigma,\tau \in \mathcal{P}_n(k)$

  $$\mathcal{M}_k \succcurlyeq 0$$
- Other symmetries

## Vector program

$$v_k(H) \coloneqq \max \sum_{\phi \in \mathcal{P}_n(2k)} c(\phi)\langle\phi\rangle$$

*Variables:* $\{|\tau\rangle \in \mathbb{C}^d : \tau \in \mathcal{P}_n(k)\}$

(any $d \geq |\mathcal{P}_n(k)|$)

*Constraints:*

- $\langle\tau|\tau\rangle = 1$
- $\langle\tau|\sigma\rangle = \langle\tau\sigma\rangle$

- They yield the same SDP
- Vector version is more convenient for rounding

# Parallels between MC and QMC: relaxation

## Max Cut

$$v_{MC} := \max \sum_{ij \in E} \frac{1 - \langle Z_i | Z_j \rangle}{2}$$

$$\begin{aligned} s.t. \quad & \langle Z_i | Z_i \rangle = 1 && \forall i \in V \\ & |Z_i\rangle \in \mathbb{R}^d && \forall i \in V \end{aligned}$$

## Quantum Max Cut ($\mathcal{L}_1$)

$$v_{QMC} := \max \sum_{ij \in E} \frac{1 - \langle X_i | X_j \rangle - \langle Y_i | Y_j \rangle - \langle Z_i | Z_j \rangle}{4}$$

$$\begin{aligned} s.t. \quad & \langle \tau_i | \tau_i \rangle = 1 && \forall i \in V, \tau \in \{X, Y, Z\} \\ & \langle \tau_i | \sigma_i \rangle = 0 && \forall i \in V, \tau, \sigma \in \{X, Y, Z\}, \tau \neq \sigma \\ & |\tau_i\rangle \in \mathbb{R}^d && \forall i \in V, \tau \in \{X, Y, Z\} \end{aligned}$$

$$v_{SQMC} := \max \sum_{ij \in E} \frac{1 - 3\langle W_i | W_j \rangle}{4}$$

$$\begin{aligned} s.t. \quad & \langle W_i | W_j \rangle = 1 && \forall i \in V \\ & |W_i\rangle \in \mathbb{R}^d && \forall i \in V \end{aligned}$$

# Parallels between MC and QMC: rounding algorithms

## Max Cut

Input: $|Z_i\rangle \in \mathbb{R}^d$ for each $i \in V$

1. Sample $|r\rangle \sim \mathcal{N}(0, I)$
2. Output $u_i \coloneqq \mathrm{sgn}(\langle Z_i | r\rangle) \in \{\pm 1\}$

Goemans-Williamson

## Quantum Max Cut ($\mathcal{L}_1$)

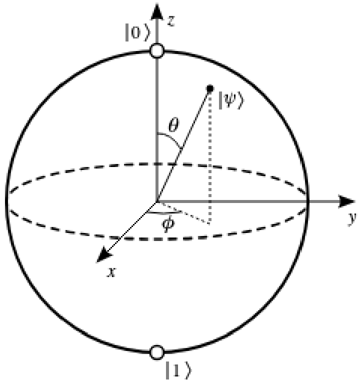
Input: $|W_i\rangle \in \mathbb{R}^d$ for each $i \in V$

1. Sample $|r_x\rangle, |r_y\rangle, |r_z\rangle \sim \mathcal{N}(0, I)$
2. Output $u_i \coloneqq \mathrm{Unit}\big(\langle W_i | r_x\rangle, \langle W_i | r_y\rangle, \langle W_i | r_z\rangle\big)$

Bloch vector

$$\rho_i = \frac{1}{2}(I + (u_i)_1 X_i + (u_i)_2 Y_i + (u_i)_3 Z_i)$$

($\rho_i$ is a pure state)




$$\rho = \bigotimes_{i \in [n]} \rho_i$$

product state

# Parallels between MC and QMC: unified analysis

## Max Cut

Input: $|Z_i\rangle \in \mathbb{R}^d$ for each $i \in V$

1. Sample $|r\rangle \sim \mathcal{N}(0, I)$
2. Output $u_i \coloneqq \mathrm{Unit}(\langle Z_i | r\rangle) \in \{\pm 1\}$

$$\mathrm{fval} = \left(1 - \langle Z_i | Z_j\rangle\right)/2$$


$$\mathrm{val} = \left(1 - u_i u_j\right)/2$$

$$\mathbb{E}_r\left[u_i u_j\right] = \mathbb{E}\left[\frac{\langle Z_i | r\rangle}{|\langle Z_i | r\rangle|} \cdot \frac{\langle Z_i | r\rangle}{|\langle Z_i | r\rangle|}\right]$$

## Quantum Max Cut ($\mathcal{L}_1$)

Input: $|W_i\rangle \in \mathbb{R}^d$ for each $i \in V$

1. Sample $|r_x\rangle, |r_y\rangle, |r_z\rangle \sim \mathcal{N}(0, I)$
2. Output $u_i \coloneqq \mathrm{Unit}\left(\langle W_i | r_x\rangle, \langle W_i | r_y\rangle, \langle W_i | r_z\rangle\right)$

$$\mathrm{fval} = \left(1 - 3\langle W_i | W_j\rangle\right)/4$$


$$\mathrm{val} = \left(1 - \langle u_i, u_j\rangle\right)/4$$

$$\mathbb{E}_{R=(r_x\, r_y\, r_z)}\left[\langle u_i, u_j\rangle\right] = \mathbb{E}\left[\left\langle \frac{R|W_i\rangle}{\|R|W_i\rangle\|}, \frac{R|W_j\rangle}{\|R|W_j\rangle\|}\right\rangle\right]$$

# Parallels between MC and QMC: unified analysis

## Max Cut

Input: $|Z_i\rangle \in \mathbb{R}^d$ for each $i \in V$

1. Sample $|r\rangle \sim \mathcal{N}(0, I)$
2. Output $u_i \coloneqq \mathrm{Unit}(\langle Z_i | r\rangle) \in \{\pm 1\}$

## Quantum Max Cut ($\mathcal{L}_1$)

Input: $|W_i\rangle \in \mathbb{R}^d$ for each $i \in V$

1. Sample $|r_x\rangle, |r_y\rangle, |r_z\rangle \sim \mathcal{N}(0, I)$
2. Output $u_i \coloneqq \mathrm{Unit}\big(\langle W_i | r_x\rangle, \langle W_i | r_y\rangle, \langle W_i | r_z\rangle\big)$

**Lemma** (Briët et al. ’14). Let $u, v$ be unit vectors in $\mathbb{R}^d$ and let $R \in \mathbb{R}^{k \times d}$ be a random matrix whose entries are *i.i.d.* Gaussian $\mathcal{N}(0,1)$. Then,

$$\mathbb{E}\left[\left\langle \frac{Ru}{\|Ru\|}, \frac{Rv}{\|Rv\|} \right\rangle\right] = F(k, \langle u, v\rangle)$$

Some hypergeometric function (explicitly computable)

# Parallels between MC and QMC: unified analysis

## Max Cut

Input: $|Z_i\rangle \in \mathbb{R}^d$ for each $i \in V$

1. Sample $|r\rangle \sim \mathcal{N}(0, I)$
2. Output $u_i \coloneqq \mathrm{Unit}(\langle Z_i|r\rangle) \in \{\pm 1\}$

$$\mathrm{fval} = \left(1 - \langle Z_i|Z_j\rangle\right)/2$$


$$\mathbb{E}_r[\mathrm{val}] = \frac{1 - F\left(1, \langle Z_i|Z_j\rangle\right)}{2}$$

## Quantum Max Cut ($\mathcal{L}_1$)

Input: $|W_i\rangle \in \mathbb{R}^d$ for each $i \in V$

1. Sample $|r_x\rangle, |r_y\rangle, |r_z\rangle \sim \mathcal{N}(0, I)$
2. Output $u_i \coloneqq \mathrm{Unit}\left(\langle W_i|r_x\rangle, \langle W_i|r_y\rangle, \langle W_i|r_z\rangle\right)$

$$\mathrm{fval} = \left(1 - 3\langle W_i|W_j\rangle\right)/4$$


$$\mathbb{E}_{r_x, r_y, r_z}[\mathrm{val}] = \frac{1 - F\left(3, \langle W_i|W_j\rangle\right)}{4}$$

# Parallels between MC and QMC: unified analysis

## Max Cut

Input: $|Z_i\rangle \in \mathbb{R}^d$ for each $i \in V$

1. Sample $|r\rangle \sim \mathcal{N}(0, I)$
2. Output $u_i \coloneqq \mathrm{Unit}(\langle Z_i|r\rangle) \in \{\pm 1\}$

$$\mathrm{fval} = \left(1 - \langle Z_i|Z_j\rangle\right)/2$$


$$\alpha = \frac{\mathbb{E}_r[\mathrm{val}]}{\mathrm{fval}} \geq \min_{t \in [-1,1]} \frac{1 - F(1,t)}{1 - t}$$

$$\approx 0.878$$

## Quantum Max Cut ($\mathcal{L}_1$)

Input: $|W_i\rangle \in \mathbb{R}^d$ for each $i \in V$

1. Sample $|r_x\rangle, |r_y\rangle, |r_z\rangle \sim \mathcal{N}(0, I)$
2. Output $u_i \coloneqq \mathrm{Unit}\left(\langle W_i|r_x\rangle, \langle W_i|r_y\rangle, \langle W_i|r_z\rangle\right)$

$$\mathrm{fval} = \left(1 - 3\langle W_i|W_j\rangle\right)/4$$


$$\alpha = \frac{\mathbb{E}_{r_x, r_y, r_z}[\mathrm{val}]}{\mathrm{fval}} \geq \min_{t \in [-1, 1/3]} \frac{1 - F(3,t)}{1 - 3t}$$

$$\approx 0.498$$

# Can we do better?

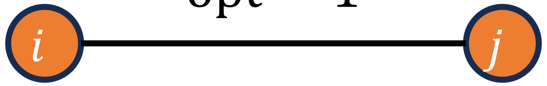
**Classical max cut:** No.

- **Khot-Kindler-Mossel-O'Donnell '07:** Assuming *Unique Game Conjecture*, it is **NP**-hard to achieve $(0.878+\epsilon)$-approximation for MC
- **Raghavendra '08, Raghavendra-Steurer '09:** Degree-2 SoS (level 1 SDP) is the best approximation algorithm for all constraints satisfaction problems (CSPs), assuming UGC

**Quantum max cut:** possible!

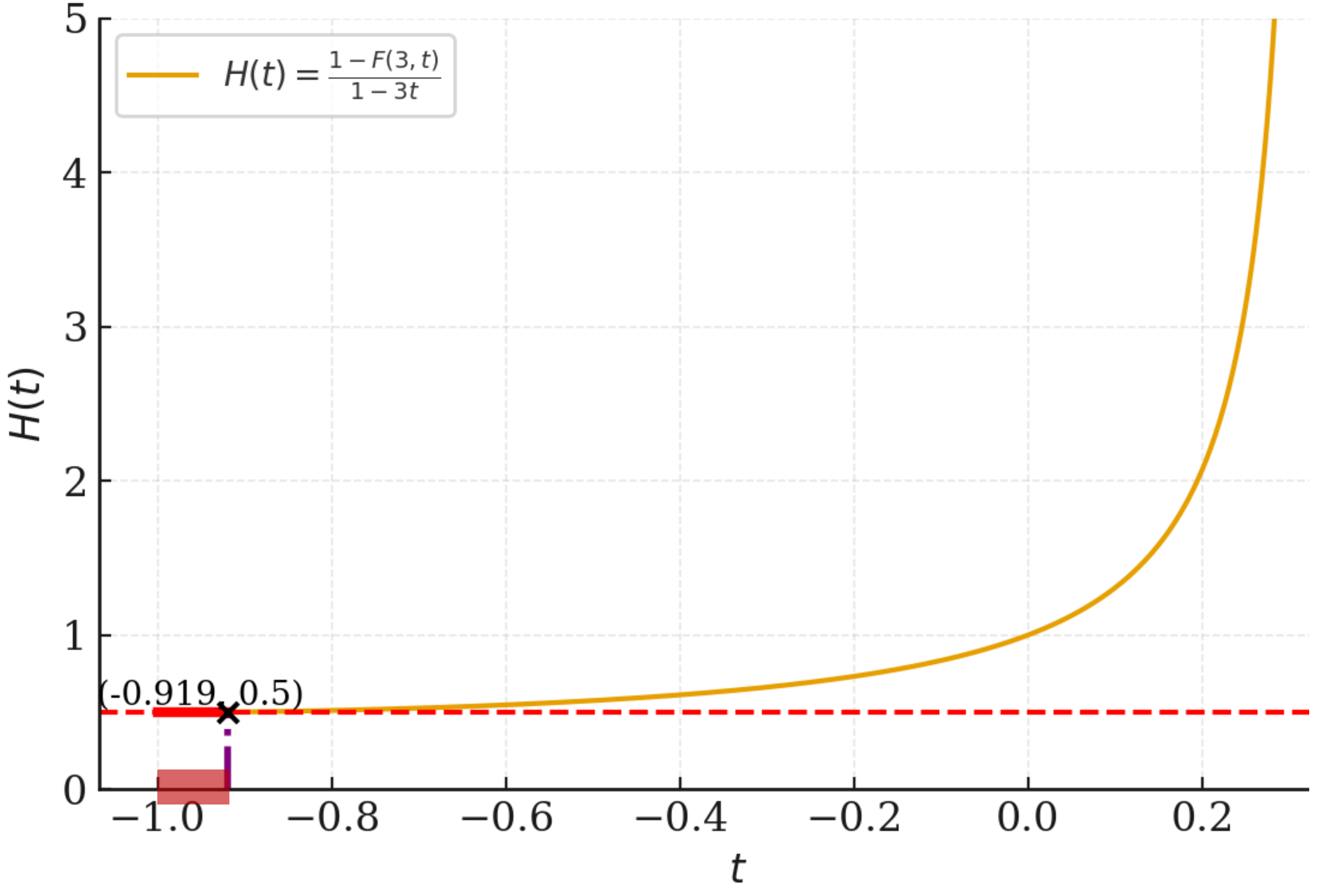
- If only using product state, the approximation ratio upper bound is $0.5 > 0.498$

$$\mathrm{opt} = 1$$




$$\mathrm{val} = \left(1 - \langle u_i, u_j \rangle\right)/4 \leq 0.5$$

# Towards better approximation

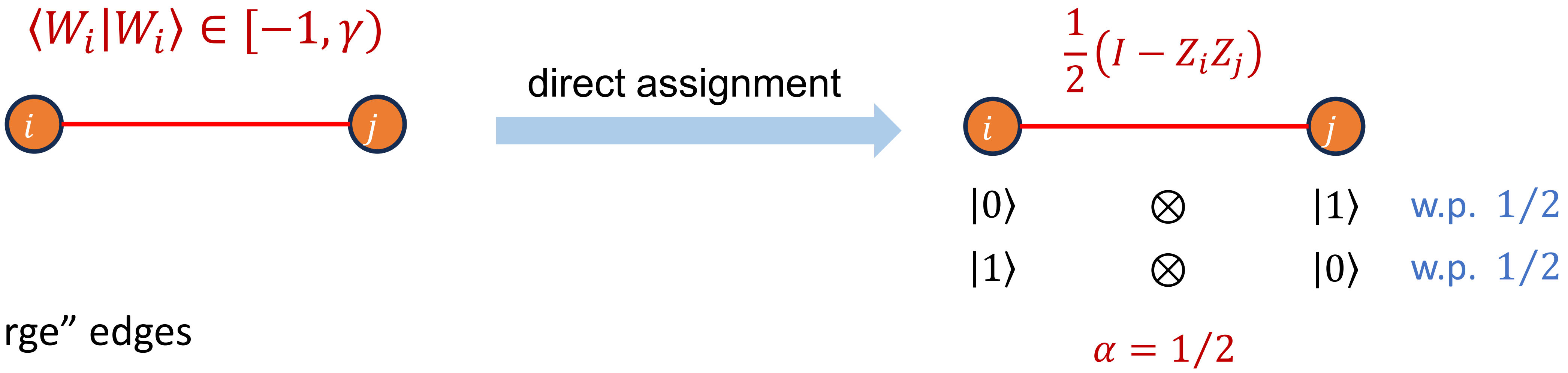



$\gamma \coloneqq -0.919 \ldots$ is a critical point

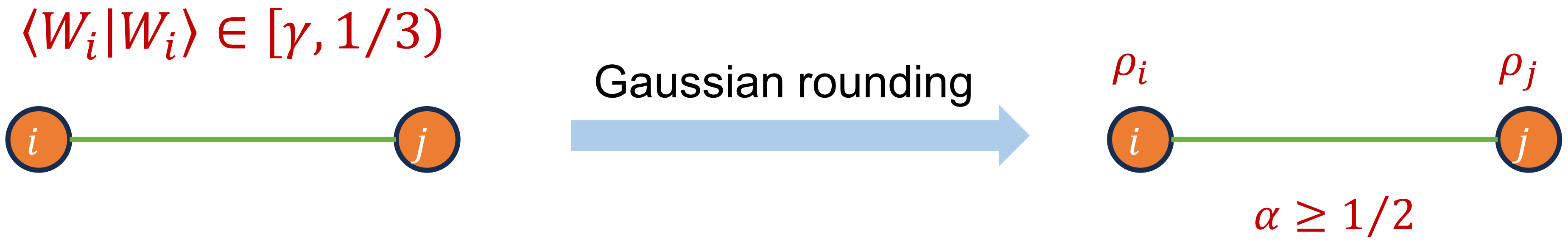
Edges with $\langle W_i | W_i \rangle > \gamma$ have approximation ratio $\geq \frac{1}{2}$

# Partial rounding

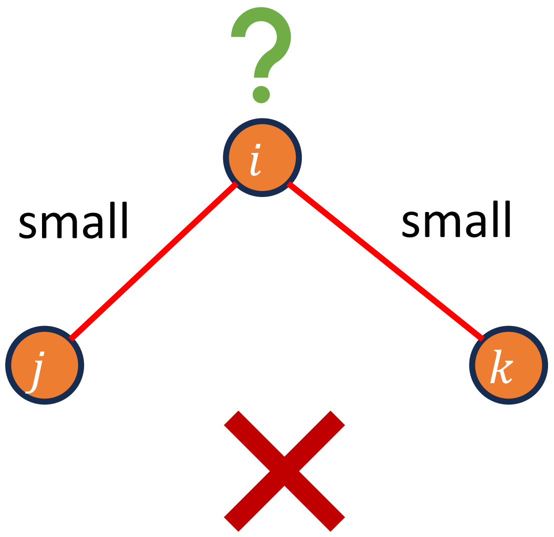
**Case 1:** "small" edges

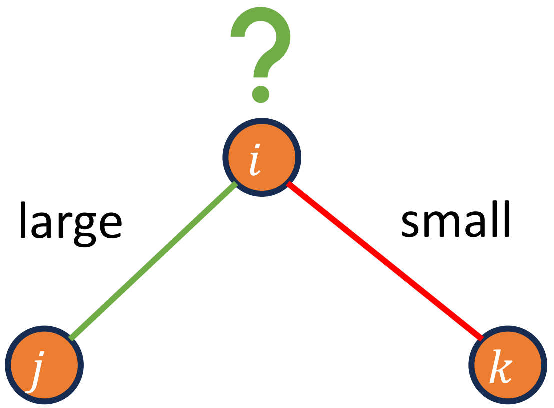



**Case 2:** "large" edges



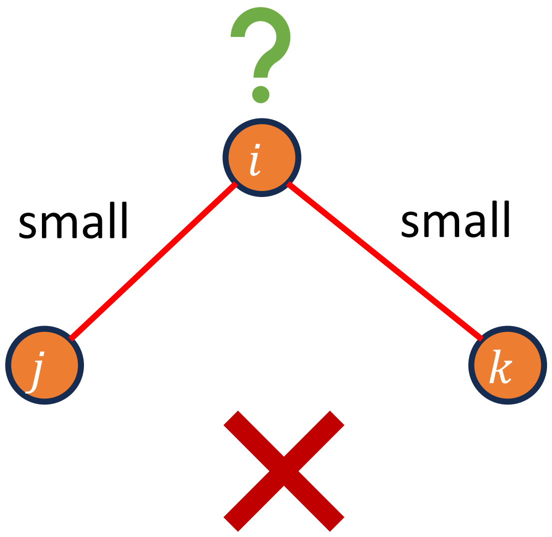
# Conflict vertices




**Fact.** The small edges form a matching in the graph

# Conflict vertices

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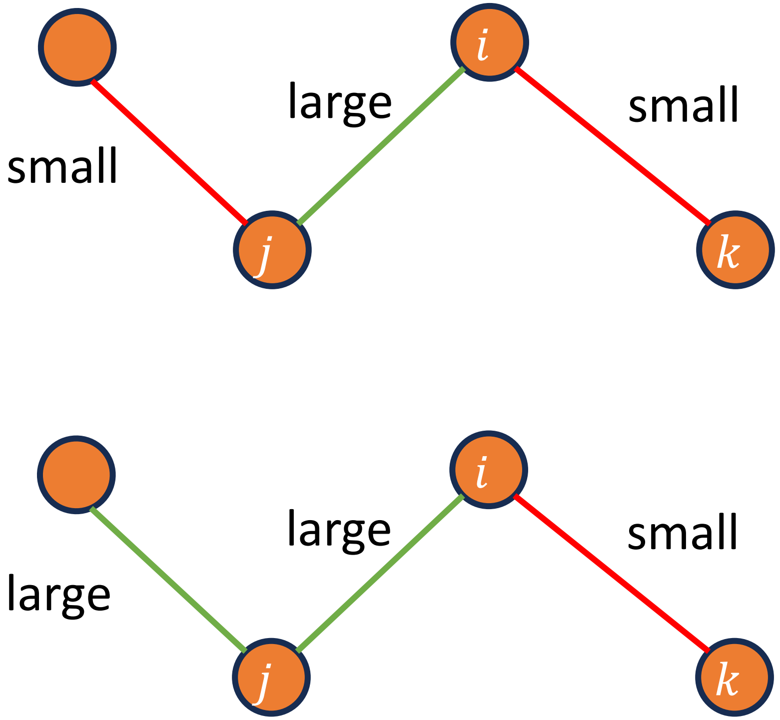
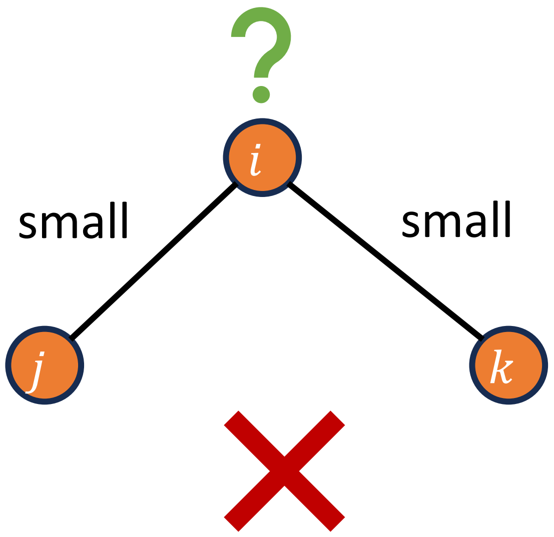
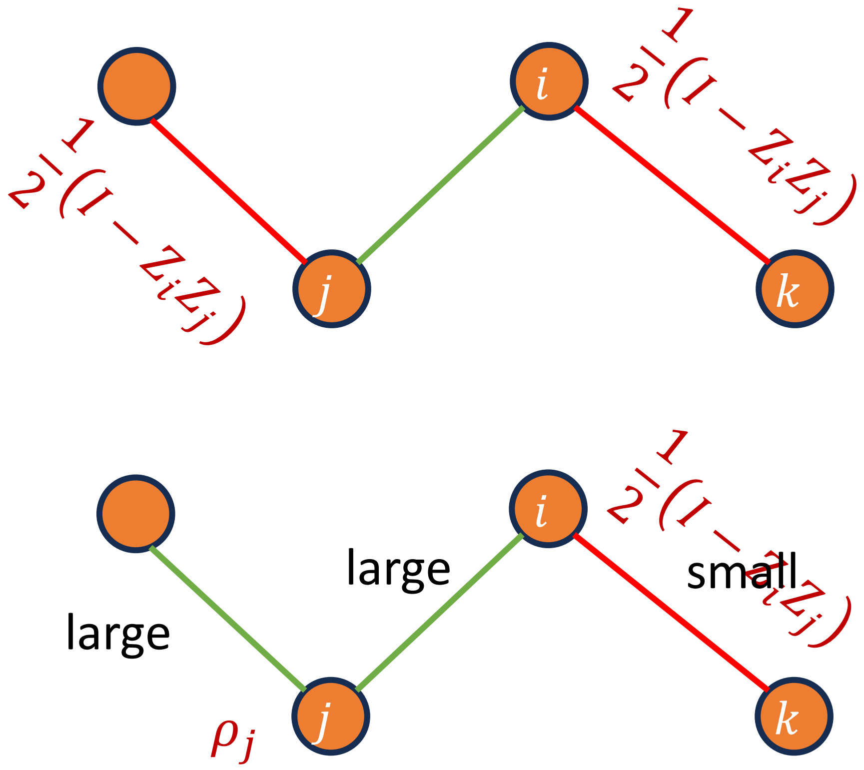


# Conflict vertices

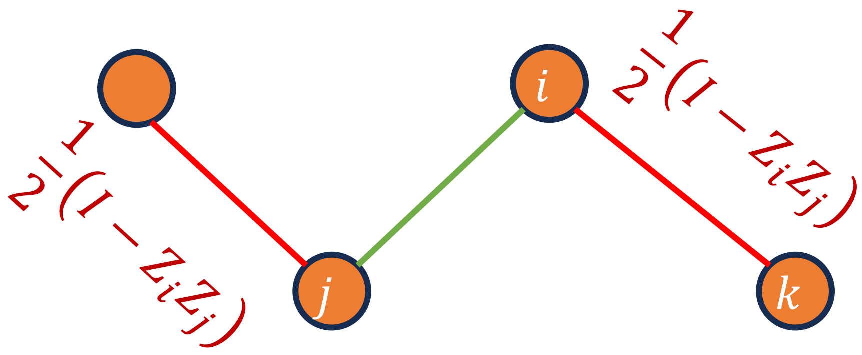


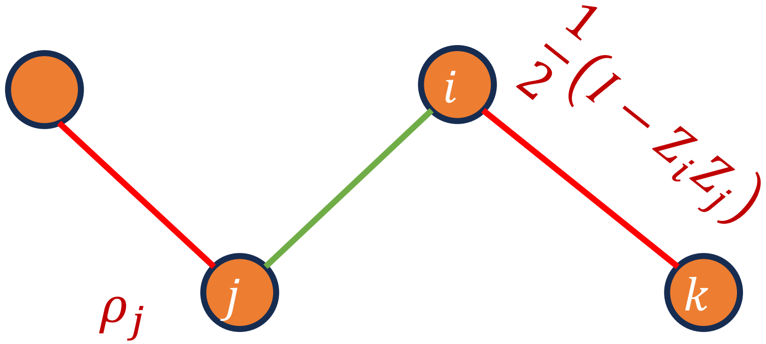

**Fact.** The small edges form a matching in the graph




We round a vertex only it does not adjacent to any small edge

# Conflict vertices




We round a large edge only if it does not intersect with any small edge

- Two qubits $i$ and $j$ are not entangled
- $\rho_i = I$ (i.e. $|0\rangle$ w.p. 0.5 and $|1\rangle$ w.p. 0.5)
- Then, no matter what $\rho_j$ is,

$$\text{val} = \text{tr}\left[h_{ij}\rho_i \otimes \rho_j\right] = 1/4$$

We need to show that the SDP value $\text{fval} < 1/2$ for this edge:

**Lemma.** If an edge $ik$ has $\langle W_i | W_k \rangle < \gamma$, then any adjacent edge $ij$ has $\langle W_i | W_j \rangle > -1/3$

- $\text{fval} = \left(1 - 3\langle W_i | W_j \rangle\right)/4 \ < 1/2$
- The small edges form a matching

# Full algorithm

1. Solve the level 2 quantum Lasserre hierarchy $\mathcal{L}_2$ for $H$
2. Let $S$ be the set of small edges
3. Let $B \coloneqq \{i \in V : \forall j, ij \notin S\}$
4. Gaussian rounding for $i \in B$, and let $\rho_i$ be the resulting state
5. Output the state

$$\rho = \bigotimes_{ij \in L} \left( \frac{I - Z_i Z_j}{4} \right) \bigotimes_{k \in \mathrm{B}} \rho_k$$

(mixed) product state

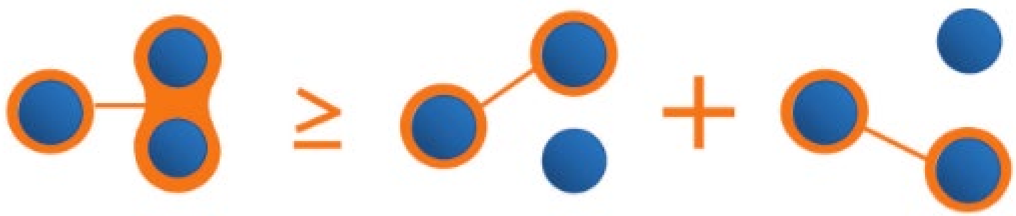
**Theorem** (Parekh-Thompson '22).

There exists an efficient classical algorithm that approximates QMC using product states and achieves the optimal $1/2$-approximation ratio

# Proof of the key lemma

**Lemma.** If an edge $ij$ has $\langle W_i | W_j \rangle < \gamma$, then any adjacent edge $ik$ has $\langle W_i | W_k \rangle > -1/3$

- Physical intuition: Monogamy of entanglement



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**Definition 17** (Quantum Lasserre hierarchy)**.** We are given as input $H = \sum_{\phi \in \mathcal{P}_n(2k)} c(\phi)\phi$, with $H \in \mathcal{H}_n$. Level $k$ of the Quantum Lasserre hierarchy, denoted $\mathcal{L}_k$, is defined by the following vector program:

$$\nu_k(H) := \max \sum_{\phi \in \mathcal{P}_n(2k)} c(\phi) \langle \phi \rangle$$

$$\text{s.t.} \quad \langle \tau | \tau \rangle = 1 \qquad \forall \tau \in \mathcal{P}_n(k) \tag{19}$$

$$\langle \tau | \sigma \rangle = \langle \tau \sigma \rangle \qquad \forall \tau, \sigma \in \mathcal{P}_n(k) \tag{20}$$

$$|\tau\rangle \in \mathbb{C}^d \qquad \forall \tau \in \mathcal{P}_n(k),$$

for any integer $d \geq |\mathcal{P}_n(k)|$.

- Unitary SWAP operator: $S_{ij} = (I + X_i X_j + Y_i Y_j + Z_i Z_j)/2$
- $\langle W_i | W_j \rangle = (\langle X_i X_j \rangle + \langle Y_i Y_j \rangle + \langle Z_i Z_j \rangle)/3 = (2\langle S_{ij} \rangle - 1)/3$

# Proof of the key lemma

**Lemma.** If an edge $ij$ has $\langle W_i|W_j\rangle < \gamma$, then any adjacent edge $ik$ has $\langle W_i|W_k\rangle > -1/3$

- Unitary SWAP operator: $S_{ij} = (I + X_iX_j + Y_iY_j + Z_iZ_j)/2$
- $\langle W_i|W_j\rangle = (\langle X_iX_j\rangle + \langle Y_iY_j\rangle + \langle Z_iZ_j\rangle)/3 = (2\langle S_{ij}\rangle - 1)/3$
- $\langle W_i|W_j\rangle < \gamma$ is equivalent to $\langle S_{ij}\rangle < (3\gamma + 1)/2$
- $\langle W_i|W_k\rangle > -1/3$ is equivalent to $\langle S_{ik}\rangle > 0$

If $\langle S_{ij}\rangle < (3\gamma + 1)/2$, then any adjacent edge $ik$ has $\langle S_{ik}\rangle > 0$

# Proof of the key lemma

Consider the Gram matrix of $|I\rangle, |S_{12}\rangle, |S_{13}\rangle, |S_{23}\rangle$:

$$\widetilde{M}(P,Q) \coloneqq \langle P|Q\rangle \quad \forall\, P,Q \in \{I, S_{12}, S_{13}, S_{23}\}$$

degree-4

- By the PSD constraint in $\mathcal{L}_2$, $\widetilde{M} \succcurlyeq 0$
- Define $M \coloneqq \left(\widetilde{M} + \widetilde{M}^\top\right)/2 \in \mathbb{R}^{4\times 4}$, $M \succcurlyeq 0$
- $M(P,P) = (\langle P^2\rangle + \langle P^2\rangle)/2 = \langle I\rangle = 1$
- $M\left(S_{ij}, I\right) = \left\langle S_{ij}\right\rangle$
- $M\left(S_{ij}, S_{ik}\right) = \left(\left\langle S_{ij}\right\rangle + \langle S_{ik}\rangle + \left\langle S_{jk}\right\rangle - \langle I\rangle\right)/2$

$$M = \begin{pmatrix} 1 & p & q & r \\ p & 1 & \frac{s-1}{2} & \frac{s-1}{2} \\ q & \frac{s-1}{2} & 1 & \frac{s-1}{2} \\ r & \frac{s-1}{2} & \frac{s-1}{2} & 1 \end{pmatrix}$$

$$p = \langle S_{12}\rangle,\ q = \langle S_{13}\rangle,\ r = \langle S_{23}\rangle,$$
$$s = p + q + r$$

Identity for SWAP operator:

$$S_{ij}S_{ik} + S_{ik}S_{ij} = \left(S_{ij} + S_{ik} + S_{jk} - I\right)/2$$

# Proof of the key lemma

**Claim 1.** $M \succcurlyeq 0$ is equivalent to:

$$0 \leq p + q + r \leq 3$$
$$p^2 + q^2 + r^2 + 2(p + q + r) - 2(pq + pr + qr) \leq 3$$

**Schur complement:**

For any symmetric matrix $M$ of the form

$$M = \begin{pmatrix} 1 & b^\top \\ b & C \end{pmatrix}$$

$M \succcurlyeq 0$ if and only if:

1. $C \succcurlyeq 0$
2. $(I - CC^+)b = 0$
3. $1 - b^\top C^+ b \geq 0$

$$M = \begin{pmatrix} 1 & p & q & r \\ p & 1 & \frac{s-1}{2} & \frac{s-1}{2} \\ q & \frac{s-1}{2} & 1 & \frac{s-1}{2} \\ r & \frac{s-1}{2} & \frac{s-1}{2} & 1 \end{pmatrix}$$

# Proof of the key lemma

**Lemma.** If $\langle S_{ij} \rangle < (3\gamma + 1)/2$, then any adjacent edge $ik$ has $\langle S_{ik} \rangle > 0$

*Proof.*

- Solving the inequalities:

$$0 \le p + q + r \le 3$$
$$p^2 + q^2 + r^2 + 2(p + q + r) - 2(pq + pr + qr) \le 3$$

- We get that if $p \le -\sqrt{3}/2$, then $q \ge 0$ (equality attained with $p = -\sqrt{3}/2\,, q = 0, r = \sqrt{3}/2$)

- Since $(3\gamma + 1)/2 < -0.878 < -\frac{\sqrt{3}}{2} \approx -0.866$, we are done

$\blacksquare$

# Approximation algorithms for quantum max cut

| Reference | Ratio | Remark |
|---|---|---|
| [GP19] | 0.498 | General graphs, outputs product state |
| [PT22] | $1/2$ | General graphs, outputs product state |
| [HTPG24] | 0.526 | General graphs, uses SOC instead of SDP |
| [AGM20] | 0.531 | General graphs |
| [PT21] | 0.533 | General graphs |
| [Lee22] | 0.562 | General graphs |
| [LP24] | 0.595 | General graphs |
| [JKK$^+$24] | 0.599 | General graphs, improved analysis of [LP24] |
| Thm. 3.11 | 0.603 | General graphs, improved analysis of [LP24] |
| [Kin23] | 0.582 | Triangle-free graphs |
| Thm. 4.5 | 0.61383 | Triangle-free graphs |
| [Kin23] | $\frac{1}{\sqrt{2}} \approx 0.707$ | Bipartite graphs |
| Thm. 5.6 | 0.8162 | Bipartite graphs |

(Gribling-Sinjorgo-Sotirov ’25)